

# POLY-SYMPLECTIC GROUPOIDS AND POLY-POISSON STRUCTURES

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**ABSTRACT.** We introduce poly-symplectic groupoids, which are natural extensions of symplectic groupoids to the context of poly-symplectic geometry, and define poly-Poisson structures as their infinitesimal counterparts. We present equivalent descriptions of poly-Poisson structures, including one related with AV-Dirac structures. We also discuss symmetries and reduction in the setting of poly-symplectic groupoids and poly-Poisson structures, and use our viewpoint to revisit results and develop new aspects of the theory initiated in [19].

## CONTENTS

1. Introduction	1
2. Poly-symplectic groupoids	3
2.1. Poly-symplectic structures	3
2.2. Multiplicative forms and poly-symplectic groupoids	4
2.3. Infinitesimal data of poly-symplectic groupoids	5
3. Poly-Poisson structures	7
3.1. Definition	7
3.2. Poly-Poisson structures and poly-symplectic groupoids	10
3.3. Poly-symplectic foliation	12
3.4. Relation with AV-Dirac structures	13
4. Symmetries and reduction	15
4.1. Poly-Poisson actions	15
4.2. Hamiltonian actions on poly-symplectic manifolds	17
4.3. Reduction and integration	20
References	23

## 1. INTRODUCTION

Poly-symplectic structures arise in the geometric formulation of Classical Field Theories in the same way that symplectic structures appear in the Hamiltonian formalism of classical mechanics [18]. More precisely, poly-symplectic structures are  $\mathbb{R}^k$ -valued 2-form, which are closed and satisfy a nondegeneracy condition, in such a way that they coincide with usual symplectic forms when  $k = 1$ . Poly-symplectic geometry has been studied in recent years by several authors, including [2, 3, 21, 23, 31]; see also [16, 20, 22, 29, 33] for further connections with physics.

In recent work [19], D. Iglesias J.C. Marrero and M. Vaquero introduced a generalization of Poisson structure by considering the inverse structures of poly-symplectic

forms, analogous to the way Poisson structures are defined from symplectic forms. In this paper, we give a new viewpoint and study new aspects of the work in [19] by considering a slight variation of their definition of poly-Poisson structure. Our definition relies on the relationship between symplectic groupoids and Poisson manifolds [35, 11], but now in the setting of *poly-symplectic groupoids*, which are natural extensions of symplectic groupoids to poly-symplectic geometry.

Similarly to symplectic groupoids, *poly-symplectic groupoids* are defined by a poly-symplectic form on a Lie groupoid satisfying a compatibility condition, which says that the poly-symplectic form is *multiplicative* (in the sense of (2.2) below). One of the main properties of symplectic groupoids is that they are the global versions of Poisson structures (see [35, 11]), that is, the manifold of objects of a symplectic groupoid is endowed with a Poisson structure whose corresponding Lie algebroid is isomorphic to the Lie algebroid of the groupoid. Moreover, the Poisson structure is uniquely determined by the condition that the target map is a Poisson morphism. Starting with a poly-symplectic groupoid, the corresponding infinitesimal geometric structure is what we identify and call *poly-Poisson structure*. In other words, the poly-Poisson structures we introduce here relate to poly-symplectic groupoids exactly in the same way that Poisson structures relate to symplectic groupoids. A similar idea in the context of multi-symplectic geometry (see [9, 10]) is studied in [8].

The notion of  $k$ -poly-Poisson structure arising in this way is slightly less general than the one given in [19], but contains the essential examples of the theory. Moreover, for  $k = 1$ , the notion agrees with ordinary Poisson structures (in contrast with the more general definition of [19]). From our viewpoint to poly-Poisson structures, we will revisit some results in [19] and extend known facts about Poisson structures, e.g., concerning their underlying Lie algebroids and foliations. Also, following the description of Poisson structures as particular cases of Dirac structures [12], we discuss an analogous picture for poly-Poisson structures. In this case, however, Dirac structures are not enough, and we must consider AV-Courant algebroids and a suitable extension of AV-Dirac structures, as in [24].

Poly-symplectic manifolds  $M$  equipped with symmetries given by a Lie group  $\mathbb{G}$  induce, under suitable regularity conditions, a quotient poly-Poisson structure on the manifold  $M/\mathbb{G}$ . In order to find poly-symplectic groupoids integrating such quotients, we need to discuss some aspects of hamiltonian actions and Marsden-Weinstein reduction in poly-symplectic geometry, see e.g. [18, 27]. This allows us to extend some constructions in [30, 17] and [7], and show that the symmetries  $\mathbb{G}$  of an integrable poly-Poisson manifold can be lifted to hamiltonian symmetries of its integrating (source-simply-connected) poly-symplectic groupoid, and that its poly-symplectic reduction at level zero is a poly-symplectic groupoid integrating the quotient poly-Poisson structure on  $M/\mathbb{G}$ .

There are several aspects of the approach to higher Poisson structures considered in this paper that we plan to pursue in future work, including the study of normal forms (see [2, 28] and the more recent work in [16]), the geometry of the corresponding higher versions of Dirac structures, and the potential connections with Field Theory.

This paper is organized as follows: In Section 2 we introduce poly-symplectic groupoids. The key result of this section, which generalizes [8, Prop. 4.1], is Proposition 2.4, where we obtain the relation between global and infinitesimal objects. Poly-Poisson structures are defined in Section 3, where we discuss their Lie algebroid structure, the underlying foliation, together with their relation with poly-symplectic groupoids via integration. Poly-Poisson structures are illustrated with some examples from [19]. At the end we give a different way to describe poly-Poisson structures related to AV-Dirac structures [24]. Section 4 is devoted to the study of symmetries of poly-Poisson structures and hamiltonian actions on poly-symplectic manifolds, see Theorem 4.1 and Prop. 4.4. Finally, applying the hamiltonian reduction, we describe integrations of quotients of an integrable poly-Poisson manifold.

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*Notation:* Lie algebroids will be denoted by  $A \rightarrow M$ , with anchor map  $\rho : A \rightarrow TM$  and bracket  $[\cdot, \cdot]$ . For a Lie groupoid  $\mathcal{G}$  over  $M$ , the source and target maps will be  $s, t : \mathcal{G} \rightarrow M$ ,  $\epsilon : M \rightarrow \mathcal{G}$  denotes the unit map,  $\text{inv} : \mathcal{G} \rightarrow \mathcal{G}$  is the inversion map, and the groupoid multiplication is  $m : \mathcal{G}_{(2)} \rightarrow \mathcal{G}$ , where the space of composable arrows is  $\mathcal{G}_{(2)} := \mathcal{G} \times_{s,t} \mathcal{G} = \{(g, h) \in \mathcal{G} \times \mathcal{G} | t(h) = s(g)\}$ . The right and left translation on the groupoid are  $R_g, L_g$ , respectively, for  $g \in \mathcal{G}$ .

For a vector space  $V$ , we will denote by  $\oplus_{(k)} V$  the  $k$ -fold direct sum of  $V$ , or equivalently, the space  $V \otimes \mathbb{R}^k$ . On vector spaces we will use two different annihilator spaces. For a vector subspaces  $W$  of a vector space  $V$ , we will denote by  $\text{Ann}(W)$  the space of elements on  $V^*$  vanishing on  $W$ . For any subspace  $S$  of  $\oplus_{(k)} V^*$ ,  $S^\circ$  stands for the space of elements on  $V$  which annihilate the elements of  $S$ , i.e  $S^\circ = \{v \in V | \alpha(v) = 0 \text{ for all } \alpha \in S\}$ . This notation will be used, more generally, for vector bundles  $E \rightarrow M$  rather than vector spaces.

The coadjoint action  $\text{Ad}^* : \mathbb{G} \rightarrow \text{End}(\mathfrak{g}^*)$  of a Lie group  $\mathbb{G}$  on the dual of its Lie algebra  $\mathfrak{g}$  induces a diagonal coadjoint action of  $\mathbb{G}$  on the product  $\mathfrak{g}_{(k)}^*$ , and we keep the notation  $\text{Ad}^*$  to this action, i.e.,  $\text{Ad}_g^*(\zeta_1, \dots, \zeta_k) = (\text{Ad}_g^* \zeta_1, \dots, \text{Ad}_g^* \zeta_k)$ .

## 2. POLY-SYMPLECTIC GROUPOIDS

In this section we will recall the concept of poly-symplectic manifold (see e.g. [18, 19, 2]) and introduce poly-symplectic groupoids, which will guide us towards poly-Poisson structures.

**2.1. Poly-symplectic structures.** A *k*-poly-symplectic form on a manifold  $M$  is a an  $\mathbb{R}^k$ -valued differential form  $\omega \in \Omega^2(M, \mathbb{R}^k)$  which is closed and nondegenerate, in the sense that the induced bundle map

$$(2.1) \quad \omega^\flat : TM \rightarrow T^*M \otimes \mathbb{R}^k$$

is injective ( $\ker(\omega) = \{0\}$ ). Writing  $\omega$  in terms of its components,  $\omega = (\omega_1, \dots, \omega_k)$ , it is poly-symplectic if and only if each  $\omega_j \in \Omega^2(M)$  is closed and

$$\cap_{j=1}^k \ker(\omega_j) = \{0\}.$$

One way to obtain examples of poly-symplectic structures is the following. Let  $M$  be a manifold endowed with  $k$  surjective, submersion maps  $p_j : M \rightarrow M_j$ , such that  $\cap_{j=1}^k \ker(dp_j) = \{0\}$ . If each  $M_j$  is equipped with a  $l_j$ -poly-symplectic form  $\omega_j$ , then

$$\omega = (p_1^* \omega_1, \dots, p_k^* \omega_k)$$

is an  $l$ -poly-symplectic form on  $M$ , where  $l = l_1 + \dots + l_k$ . In particular, if  $(M_j, \omega_j)$  is an  $l_j$ -poly-symplectic manifold,  $j = 1, \dots, k$ , this construction endows  $M := M_1 \times \dots \times M_k$  with an  $l$ -poly-symplectic structure, for  $l = l_1 + \dots + l_k$ . This shows that the product of  $k$  symplectic manifolds naturally carries a  $k$ -poly-symplectic structure.

The following is a particular case of interest in classical field theory [18]:

**Example 2.1.** ( $k$ -covelocities on a manifold) Recall that any cotangent bundle  $T^*Q$  has a canonical symplectic form  $\omega_{can}$ . The manifold of  $k$ -covelocities is the Whitney sum

$$\oplus_{(k)} T^*Q = T^*Q \oplus \dots \oplus T^*Q,$$

which is equipped with the natural projections  $\text{pr}_j : \oplus_{(k)} T^*Q \rightarrow T^*Q$ . It is clear that  $\cap_{j=1}^k \ker(d\text{pr}_j) = \{0\}$ , and

$$\omega := (\text{pr}_1^* \omega_{can}, \dots, \text{pr}_k^* \omega_{can}) \in \Omega^2(\oplus_{(k)} T^*Q, \mathbb{R}^k)$$

is a  $k$ -poly-symplectic form.

Other examples of poly-symplectic structures are discussed e.g. in [18, 19, 32].

**2.2. Multiplicative forms and poly-symplectic groupoids.** We now consider poly-symplectic structures on Lie groupoids. Let  $\mathcal{G}$  be a Lie groupoid over  $M$ .

A differential form  $\theta \in \Omega^r(\mathcal{G})$  is called *multiplicative* if it satisfies

$$(2.2) \quad m^* \theta = \text{pr}_1^* \theta + \text{pr}_2^* \theta,$$

where  $\text{pr}_i : \mathcal{G} \times_{s,t} \mathcal{G} \rightarrow \mathcal{G}$  are the projection maps. Note that condition (2.2) still makes sense for  $\mathbb{R}^k$ -valued forms  $\theta = (\theta_1, \dots, \theta_k)$ , and it simply says that each component  $\theta_i$  is multiplicative.

Recall that a *symplectic groupoid* is a Lie groupoid  $\mathcal{G} \rightrightarrows M$  endowed with a multiplicative symplectic form  $\omega \in \Omega^2(\mathcal{G})$ , see e.g. [11, 35]. A direct generalization leads to

**Definition 2.1.** A  $k$ -poly-symplectic groupoid is a Lie groupoid  $\mathcal{G} \rightrightarrows M$  together with a  $k$ -poly-symplectic form  $\omega = (\omega_1, \dots, \omega_k) \in \Omega^2(\mathcal{G}, \mathbb{R}^k)$  satisfying (2.2). More explicitly, each  $\omega_j \in \Omega^2(\mathcal{G})$  is closed, multiplicative, and  $\cap_{j=1}^k \ker(\omega_j) = \{0\}$ .

Suppose that  $\mathcal{G}_j \rightrightarrows M_j$  are  $l_j$ -poly-symplectic groupoids,  $j = 1, \dots, k$ . As discussed in Section 2.1, we can verify that if a Lie groupoid  $\mathcal{G}$  is equipped with surjective submersions  $p_j : \mathcal{G} \rightarrow \mathcal{G}_j$ ,  $j = 1, \dots, k$ , which are *groupoid morphisms* and satisfy  $\cap_j \ker(dp_j) = \{0\}$ , then  $\omega = (p_1^* \omega_1, \dots, p_k^* \omega_k) \in \Omega^2(\mathcal{G}, \mathbb{R}^k)$  makes  $\mathcal{G}$  into an  $l$ -poly-symplectic groupoid, for  $l = l_1 + \dots + l_k$ . Here we use the fact that the pull-back of a multiplicative form by a groupoid morphism is again multiplicative. In particular, we have:

**Proposition 2.2.** The direct product of symplectic groupoids  $(\mathcal{G}_j, \omega_j)$ ,  $j = 1, \dots, k$ , naturally carries a multiplicative  $k$ -poly-symplectic structure given by

$$\omega = (\text{pr}_1^* \omega_1, \dots, \text{pr}_k^* \omega_k),$$

where  $\text{pr}_j : \mathcal{G}_1 \times \dots \times \mathcal{G}_k \rightarrow \mathcal{G}_j$  is the natural projection.

More conceptually, multiplicative poly-symplectic forms are very special cases of multiplicative forms with values in representations, as in [15]. Given a Lie groupoid  $\mathcal{G} \rightrightarrows M$  and a vector bundle  $E \rightarrow M$ , consider the pullback bundle  $t^*E \rightarrow \mathcal{G}$ . An  $E$ -valued  $r$ -form on  $\mathcal{G}$  is an element  $\theta \in \Omega^r(\mathcal{G}, t^*E)$ . If  $E$  is a representation of  $\mathcal{G}$  (see [25]), we say that  $\theta \in \Omega^r(\mathcal{G}, t^*E)$  is *multiplicative* if for all composable arrows  $(g, h) \in \mathcal{G} \times_{s,t} \mathcal{G}$  we have

$$(2.3) \quad (m^*\theta)_{(g,h)} = \text{pr}_1^*\theta + g \cdot (\text{pr}_2^*\theta),$$

where  $m, \text{pr}_1, \text{pr}_2$  are as in (2.2). It is clear that for the trivial bundle  $E = \mathbb{R}^k \times M$ , equipped with the trivial representation, this recovers the notion of multiplicative  $\mathbb{R}^k$ -valued forms previously discussed.

For later use, we observe the  $E$ -valued version of the equations in [4, Lemma 3.1(i)]:

**Lemma 2.3.** *If  $\theta \in \Omega^k(\mathcal{G}, t^*E)$  is multiplicative then*

$$(2.4) \quad \epsilon^*\theta = 0, \quad \text{and} \quad \theta_g = -g \cdot (\text{inv}^*\theta_{\text{inv}(g)})$$

for all  $g \in \mathcal{G}$ .

*Proof.* Define the map  $(\text{Id} \times \text{inv})(g) := (g, g^{-1})$  from  $\mathcal{G}$  to  $\mathcal{G}_{(2)}$ . If we apply the pull-back of  $(\text{Id} \times \text{inv})$  to Equation (2.3) and recall that  $\epsilon \circ t = m \circ (\text{Id} \times \text{inv})$ , we obtain:

$$\begin{aligned} t^*\epsilon^*\theta_{\epsilon(t(g))} &= (\text{Id} \times \text{inv})^*(m^*\theta)_{(g,g^{-1})} = \theta_g + (\text{Id} \times \text{inv})^*(g \cdot (\text{pr}_2^*\theta_{g^{-1}})) \\ &= \theta_g + g \cdot ((\text{Id} \times \text{inv})^*\text{pr}_2^*\theta_{g^{-1}}). \end{aligned}$$

Therefore

$$(2.5) \quad t^*\epsilon^*\theta_{\epsilon(t(g))} = \theta_g + g \cdot (\text{inv}^*\theta_{g^{-1}}).$$

If in particular we fix  $g = \epsilon(m)$  for some  $m \in M$  and take the pull-back by the unit map in (2.5), we conclude that  $\epsilon^*\theta = 0$ . Using this identity and (2.5), it follows that  $\theta_g + g \cdot (\text{inv}^*\theta_{g^{-1}}) = 0$ .  $\square$

**2.3. Infinitesimal data of poly-symplectic groupoids.** It is well known that Poisson structures are the infinitesimal counterparts of symplectic groupoids, see e.g. [11, 35]. We will now discuss the infinitesimal counterpart of poly-symplectic groupoids, in the spirit of [8], which leads to a generalization of Poisson structures in poly-symplectic geometry.

Let  $A \rightarrow M$  denote the Lie algebroid of a Lie groupoid  $\mathcal{G} \rightrightarrows M$ , with anchor  $\rho : A \rightarrow TM$  and bracket  $[\cdot, \cdot]$  on  $\Gamma(A)$ . Recall from [1, 6, 4] that a closed multiplicative  $r$ -form  $\theta$  on  $\mathcal{G}$  is infinitesimally described by a bundle map (over the identity)

$$\mu : A \rightarrow \wedge^{r-1} T^*M,$$

satisfying the conditions

$$(2.6) \quad i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u), \quad \forall u, v \in A$$

$$(2.7) \quad \mu([u, v]) = \mathcal{L}_{\rho(u)}\mu(v) - i_{\rho(v)}d\mu(u), \quad \forall u, v \in \Gamma(A).$$

The map  $\mu$  is related to  $\theta$  via

$$(2.8) \quad i_{uR}\theta = t^*(\mu(u)),$$

for  $u \in \Gamma(A)$ , where  $u^R$  denotes the right-invariant vector field on  $\mathcal{G}$  defined by  $u$ . For source-simply-connected Lie groupoids,  $\mu$  and  $\theta$  completely determine one another.

It follows that a closed multiplicative  $\mathbb{R}^k$ -valued 2-form  $\omega = (\omega_1, \dots, \omega_k) \in \Omega^2(\mathcal{G}, \mathbb{R}^k)$  infinitesimally corresponds to a bundle map

$$(2.9) \quad \mu = (\mu_1, \dots, \mu_k) : A \rightarrow \oplus_{(k)} T^*M$$

satisfying the same equations (2.6) and (2.7), which simply means the equations are satisfied componentwise, i.e., each  $\mu_j : A \rightarrow T^*M$  is a closed IM 2-form. For the complete infinitesimal description of a multiplicative poly-symplectic form, it remains to express the non-degeneracy condition  $\cap_{j=1}^k \ker(\omega_j) = \{0\}$  in terms of the map  $\mu$  in (2.9). We will do that in the more general framework of multiplicative forms on  $\mathcal{G}$  with values in representations  $E \rightarrow M$ .

The infinitesimal version of multiplicative  $E$ -valued  $r$ -forms on a Lie groupoid  $\mathcal{G}$  was studied in [15], where it is proven that (under the usual source-simply-connectedness condition on  $\mathcal{G}$ ) such forms  $\theta$  are in 1-1 correspondence with pairs of maps  $(D, \mu)$ ,

$$D : \Gamma(A) \rightarrow \Omega^r(M, E), \quad \mu : A \rightarrow \wedge^{r-1} T^*M \otimes E,$$

satisfying suitable conditions (that we will not need explicitly), see [15, Sec. 2.2]. We will only need the following facts about the infinitesimal data  $(D, \mu)$ . First, the relation between the bundle map  $\mu$  and the multiplicative  $E$ -valued form  $\theta$  is a direct generalization of that in (2.8): indeed, using [4, Eqs. (3.1)-(3.3)], it follows that the second equation of [15, (2.4)] is equivalent to

$$(2.10) \quad i_{u^R} \theta = t^*(\mu(u)).$$

Second, when  $E = \mathbb{R}^k$  is the *trivial* representation and the multiplicative form  $\theta$  is *closed*, then  $D$  is determined by  $\mu$ , in fact  $D = d\mu$  (see [6]); so in this case one only needs  $\mu$  for the infinitesimal description of  $\theta$ .

We say that an  $r$ -form  $\theta \in \Omega^r(\mathcal{G}, t^*E)$  is *non-degenerate* when the map

$$\theta^\flat : T\mathcal{G} \rightarrow \wedge^{r-1} T^*\mathcal{G} \otimes t^*E, \quad X \mapsto i_X \theta$$

has trivial kernel. When  $\theta$  is multiplicative, we have the following infinitesimal description of this property.

**Proposition 2.4.** *Consider  $\theta \in \Omega^r(\mathcal{G}, t^*E)$  a multiplicative  $E$ -valued  $r$ -form on a Lie groupoid  $\mathcal{G}$ , and let  $\mu : A \rightarrow \wedge^{r-1} T^*M \otimes E$  be such that (2.10) holds. Then  $\theta$  is nondegenerate if and only if*

$$(2.11) \quad \ker(\mu) = \{0\}, \quad \text{and} \quad (\text{Im}(\mu))^\circ = \{0\},$$

where  $(\text{Im}(\mu))^\circ = \{X \in TM \mid i_X \mu(u) = 0 \text{ for all } u \in A\}$ .

*Proof.* The proof uses the relation (2.10) and follows the same idea of [8, Prop. 4.1]. We recall the details for the reader's convenience.

First we suppose that conditions (2.11) hold for  $\mu$  and take  $X \in T_g \mathcal{G}$  in the kernel of the multiplicative form. We get that  $dtX = 0$  because  $i_X t^*(\mu(u)) = 0$  for all  $u \in A$  (from (2.10)), hence  $X$  is tangent to the  $t$ -fibers, which implies the existence of  $v \in A$  for which  $X = v_g^L = d_g \text{inv}(v_g^R)$ . As consequence of the second equation in (2.4) and (2.10), we see that  $-g \cdot (s^*(\mu(v))) = i_{v_g^L} \theta_g = i_X \theta_g = 0$  for any  $g \in \mathcal{G}$ , hence  $s^*(\mu(v)) = 0$ . This shows that  $v \in \ker(\mu)$ , therefore  $X = v_g^L = 0$ .

For the other direction, let  $u \in \ker(\mu)$ . Then  $i_{u^R}\theta = t^*(\mu(u)) = 0$ , which implies  $u^R = 0$  by nondegeneracy of the form, thus the first condition in (2.11) holds. Now fixing  $X \in (\text{Im}(\mu))_m^\circ$  for  $m \in M$ , (2.10) implies that  $i_u i_X \theta = 0$  for all  $u \in A_m$ . The splitting  $T_m \mathcal{G} = T_m M \oplus A_m$  allows us to write  $Z_j = X_j + u_j \in T_m \mathcal{G}$ ,  $j = 1, \dots, r-1$ , and the multilinearity of  $\theta$  implies that

$$i_{Z_{r-1}} \dots i_{Z_1} i_X \theta = i_{X_{r-1}} \dots i_{X_1} i_X \theta,$$

because the other terms vanish from the fact that  $i_u i_X \theta = 0$  for all  $u \in A_m$ . Now the first condition in (2.3) implies that  $i_{Z_{r-1}} \dots i_{Z_1} i_X \theta = 0$  for all  $Z_j \in T_m \mathcal{G}$ , hence  $X = 0$ .  $\square$

For the trivial representation  $E = M \times \mathbb{R}$  and forms of arbitrary degree  $r$ , Proposition 2.4 recovers [8, Prop. 4.1]. For the trivial representation  $E = M \times \mathbb{R}^k$  and  $r = 2$ , we obtain the infinitesimal description of multiplicative  $k$ -poly-symplectic forms.

**Corollary 2.5.** *Let  $\mathcal{G} \rightrightarrows M$  be a source-simply-connected groupoid. Then there is a one-to-one correspondence between multiplicative poly-symplectic forms  $\omega \in \Omega^2(\mathcal{G}, \mathbb{R}^k)$  and bundle maps  $\mu : A \rightarrow \oplus_{(k)} T^*M$  satisfying (2.6), (2.7) and (2.11) via the relation  $i_{u^R}\omega = t^*(\mu(u))$ , for all  $u \in \Gamma(A)$ .*

Given a Lie algebroid  $A \rightarrow M$ , we see from Corollary 2.5 that bundle maps  $\mu : A \rightarrow \oplus_{(k)} T^*M$  satisfying (2.6), (2.7) and (2.11) are the infinitesimal counterparts of multiplicative poly-symplectic forms on Lie groupoids. So we refer to these objects as *IM poly-symplectic forms*, where “IM” stands for “infinitesimally multiplicative”. We say that two IM poly-symplectic forms  $\mu : A \rightarrow \oplus_{(k)} T^*M$  and  $\mu' : A' \rightarrow \oplus_{(k)} T^*M$  are *equivalent* if there is a Lie algebroid isomorphism  $\varphi : A \rightarrow A'$  such that  $\mu = \mu' \circ \varphi$ . Under the equivalence in Corollary 2.5, they correspond to isomorphic poly-symplectic groupoids.

We will now use the infinitesimal geometry of poly-symplectic groupoids described in Corollary 2.5 to provide a new viewpoint to [19].

### 3. POLY-POISSON STRUCTURES

**3.1. Definition.** The notion of *poly-Poisson structure* that we now introduce is a slight modification of that in [19].

**Definition 3.1.** *A  $k$ -poly-Poisson structure on a manifold  $M$  is a pair  $(S, P)$ , where  $S \rightarrow M$  is a vector subbundle of  $\oplus_{(k)} T^*M$  and  $P : S \rightarrow TM$  is a vector-bundle morphism (over the identity) such that*

- (i)  $i_{P(\bar{\eta})}\bar{\eta} = 0$ , for all  $\bar{\eta} \in S$ ,
- (ii)  $S^\circ = \{X \in TM \mid i_X \bar{\eta} = 0, \forall \bar{\eta} \in S\} = \{0\}$ ,
- (iii) *the space of section  $\Gamma(S)$  is closed under the bracket*

$$(3.1) \quad [\bar{\eta}, \bar{\gamma}] := \mathcal{L}_{P(\bar{\eta})}\bar{\gamma} - \mathcal{L}_{P(\bar{\gamma})}\bar{\eta} + d(i_{P(\bar{\gamma})}\bar{\eta}) = \mathcal{L}_{P(\bar{\eta})}\bar{\gamma} - i_{P(\bar{\gamma})}d\bar{\eta}, \quad \text{for } \bar{\gamma}, \bar{\eta} \in \Gamma(S),$$

*and the restriction of this bracket to  $\Gamma(S)$  satisfies the Jacobi identity.*

We will call the triple  $(M, S, P)$  a  *$k$ -poly-Poisson manifold*.

We observe that the bracket (3.1) is skew-symmetric (by condition (i)) and satisfies the Leibniz rule:

$$[\bar{\eta}, f\bar{\gamma}] = f[\bar{\eta}, \bar{\gamma}] + (\mathcal{L}_{P(\bar{\eta})}f)\bar{\gamma},$$

for all  $\bar{\eta}, \bar{\gamma} \in \Gamma(S)$  and  $f \in C^\infty(M)$ . It follows that, for a poly-Poisson manifold  $(M, S, P)$ , the vector bundle  $S \rightarrow M$  is a *Lie algebroid* with bracket (3.1) and anchor map  $P : S \rightarrow TM$ . Since for any Lie algebroid the anchor map preserves Lie brackets, we have that

$$(3.2) \quad P([\bar{\eta}, \bar{\gamma}]) = [P(\bar{\eta}), P(\bar{\gamma})], \quad \forall \bar{\eta}, \bar{\gamma} \in \Gamma(S).$$

**Remark 3.2.** *In (iii) of Def. 3.1, assuming that  $\Gamma(S)$  is closed under the bracket (3.1), we can replace the condition on the Jacobi identity by the bracket-preserving property (3.2). Indeed, if (3.2) holds and for  $\bar{\eta}, \bar{\lambda}, \bar{\gamma} \in \Gamma(S)$ , then*

$$\begin{aligned} [[\bar{\eta}, \bar{\gamma}], \bar{\lambda}] + [\bar{\gamma}, [\bar{\eta}, \bar{\lambda}]] &= \mathcal{L}_{[P(\bar{\eta}), P(\bar{\gamma})]} \bar{\lambda} - i_{P(\bar{\lambda})} d[\bar{\eta}, \bar{\lambda}] + \mathcal{L}_{P(\bar{\gamma})} [\bar{\eta}, \bar{\lambda}] - i_{[P(\bar{\eta}), P(\bar{\lambda})]} d\bar{\gamma} \\ &= \mathcal{L}_{P(\bar{\eta})} \mathcal{L}_{P(\bar{\gamma})} \bar{\lambda} - i_{P(\bar{\lambda})} \mathcal{L}_{P(\bar{\eta})} d\bar{\gamma} + i_{P(\bar{\lambda})} \mathcal{L}_{P(\bar{\gamma})} d\bar{\eta} - \mathcal{L}_{P(\bar{\gamma})} i_{P(\bar{\lambda})} d\bar{\eta} - i_{[P(\bar{\eta}), P(\bar{\lambda})]} d\bar{\gamma} \\ &= \mathcal{L}_{P(\bar{\eta})} \mathcal{L}_{P(\bar{\gamma})} \bar{\lambda} - \mathcal{L}_{P(\bar{\eta})} i_{P(\bar{\lambda})} d\bar{\gamma} + i_{P(\bar{\lambda})} \mathcal{L}_{P(\bar{\gamma})} d\bar{\eta} - \mathcal{L}_{P(\bar{\gamma})} i_{P(\bar{\lambda})} d\bar{\eta} \\ &= \mathcal{L}_{P(\bar{\eta})} [\bar{\gamma}, \bar{\lambda}] - i_{[P(\bar{\gamma}), P(\bar{\lambda})]} d\bar{\eta} = [\bar{\eta}, [\bar{\gamma}, \bar{\lambda}]], \end{aligned}$$

where the second equality holds by  $\mathcal{L}_{[X, Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$  and the third results from Cartan's magic formula.

It follows from this remark that condition (iii) in Def. 3.1 is equivalent to

(iii)' *the space of section  $\Gamma(S)$  is closed under the bracket (3.1) and (3.2) holds.*

**Remark 3.3** (Comparison with [19]). *The notion of poly-Poisson structure in Def. 3.1 is slightly more restrictive than the notion introduced by Iglesias, Marrero and Vaquero in [19, Def. 3.1]. The difference is that in [19] our condition (ii) in Def. 3.1, namely  $S^\circ = \{0\}$ , is replaced by the following weaker requirement:*

$$(3.3) \quad \text{Im}(P) \cap S^\circ = \{0\}.$$

We will refer to such objects as weak-poly-Poisson structures.

Let  $(M_j, S_j, P_j)$ ,  $j = 1, 2$ , be  $k$ -poly-Poisson manifolds.

**Definition 3.4.** *A smooth map  $f : M_1 \rightarrow M_2$  is called a poly-Poisson morphism if*

- a)  $f^* \bar{\eta} \in S_1$  for all  $\bar{\eta} \in S_2$ ,
- b) for every  $x \in M_1$  and  $\bar{\eta} \in S_2|_{f(x)}$ ,  $Tf|_x(P_1(Tf|_x^* \bar{\eta})) = P_2(\bar{\eta})$ .

The following are basic examples of Def. 3.1.

**Example 3.1.** For  $k = 1$ , a  $k$ -poly-Poisson structure is simply a usual Poisson structure. Indeed, if  $S$  is subbundle of  $T^*M$ , condition (ii) in Def. 3.1 shows that

$$S = T^*M.$$

(Note that this is not guaranteed by the weaker condition (3.3).) Condition (i) shows that  $P : T^*M \rightarrow TM$  is of the form  $P = \pi^\sharp$  for a bivector field  $\pi \in \Gamma(\wedge^2 TM)$ , where  $\pi^\sharp(\alpha) = i_\alpha \pi$ . Finally, condition (iii) amounts to the usual integrability condition  $[\pi, \pi] = 0$  (i.e., the bracket on  $C^\infty(M)$  given by  $(f, g) \mapsto \pi(df, dg)$  satisfies the Jacobi identity). The Lie algebroid structure on  $S = T^*M$  is the usual one for Poisson manifolds [34]: the anchor is  $\pi^\sharp$  and the bracket  $[\cdot, \cdot]$  on  $\Omega^1(M)$  is the one such that  $[df, dg] = d(\pi(df, dg))$ . The notion of morphism in Def. 3.4 also recovers to the usual notion of Poisson morphism.



**Example 3.2.** Let  $(M, \omega)$  be a  $k$ -poly-symplectic manifold, and consider the injective bundle map  $\omega^\flat : TM \rightarrow \oplus_{(k)} T^*M$ . We define a subbundle  $S_\omega$  of  $\oplus_{(k)} T^*M$  and a bundle map  $P_\omega : S \rightarrow TM$  as follows:

$$(3.4) \quad S_\omega := \text{Im}(\omega^\flat) \quad \text{and} \quad P_\omega(i_X \omega) := X \in TM.$$

See [19, Prop. 2.3 and Example 3.3]. Note that condition (ii) in Def. 3.1 is equivalent to the non-degeneracy of  $\omega$ .

Moreover, given  $k$ -poly-symplectic manifolds  $(M_j, \omega_j)$ ,  $j = 1, 2$ , a diffeomorphism  $f : M_1 \rightarrow M_2$  preserves poly-Poisson structures (as in Def. 3.4) if and only if

$$f^* \omega_2 = \omega_1.$$

**Example 3.3.** Let  $Q$  be a manifold. We can always regard it as a Poisson manifold with the Poisson bracket that is identically zero. For each  $k$ , we can also view  $Q$  as a  $k$ -poly-Poisson manifold, and this can be done in several ways. For example,  $S_1 = \oplus_{(k)} T^*Q$  and  $P_1 = 0$  define a poly-Poisson structure on  $Q$ , and the same is true for  $S_2 = \{\alpha \oplus \dots \oplus \alpha \mid \alpha \in T^*Q\} \subset \oplus_{(k)} T^*Q$  and  $P_2 = 0$ , or  $S_3 = \{\alpha \oplus 0 \oplus \dots \oplus 0 \mid \alpha \in T^*Q\} \subset \oplus_{(k)} T^*Q$  and  $P_3 = 0$ .

Considering  $\oplus_{(k)} T^*Q$  equipped with its poly-symplectic structure (see Example 2.1), the natural projection  $\oplus_{(k)} T^*Q \rightarrow Q$  is a poly-Poisson map when  $Q$  is equipped with either one of the poly-Poisson structures  $(S_i, P_i)$ , for  $i = 1, 2, 3$ .

**Remark 3.5.** *It is a well-known fact in Poisson geometry that  $M$  is a Poisson manifold and  $f : M \rightarrow N$  is a surjective submersion, then there is at most one Poisson structure on  $N$  for which  $f$  is a Poisson map. Example 3.3 shows that this is not necessarily the case for  $k$ -poly-Poisson structures, for  $k \geq 2$ .*

*On the other hand, let  $M$  be a  $k$ -poly-Poisson manifold and  $f : M \rightarrow N$  be a surjective submersion. Then if  $(S_1, P_1)$  and  $(S_2, P_2)$  are  $k$ -poly-Poisson structures on  $N$  for which  $f$  is a poly-Poisson map and we know that  $S_1 = S_2$ , then  $P_1 = P_2$ .*

As explained in [19, Example 3.8], the product of Poisson manifolds carries a natural poly-Poisson structure.

**Example 3.4.** Let  $(M_j, \pi_j)$ ,  $j = 1, \dots, k$ , be Poisson manifolds. Let  $M = M_1 \times \dots \times M_k$ . Denote by  $S_j$  the natural inclusion of  $T^*M_j$  into  $T^*M$ , and let  $S \subset \oplus_{(k)} T^*M$  be defined by  $S := S_1 \oplus \dots \oplus S_k$ . Consider the bundle map  $P : S \rightarrow TM$ ,

$$P(\alpha_1, \dots, \alpha_k) = (\pi_1^\sharp(\alpha_1), \dots, \pi_k^\sharp(\alpha_k)),$$

where  $\alpha_j \in S_j$ . One may verify that  $(M, S, P)$  is a  $k$ -poly-Poisson manifold directly from the definition.

In addition, let  $f_j : (M_j, \pi_j) \rightarrow (N_j, \Lambda_j)$  for  $j = 1, \dots, k$ , be  $k$  Poisson maps between the Poisson manifolds  $M_j$  and  $N_j$  respectively. From the construction above we obtain  $k$ -poly-Poisson structures  $(S_M, P_M)$  and  $(S_N, P_N)$  on the product manifolds  $M = \prod_{j=1}^k M_j$  and  $N = \prod_{j=1}^k N_j$ , and denote by  $pr_j^M : M \rightarrow M_j$ , and  $pr_j^N : N \rightarrow N_j$  the natural projections. The Poisson maps  $f_j$  induce a product map  $\bar{f} = (f_1, \dots, f_k) : M \rightarrow N$  that, as a consequence of the definition of the  $k$ -poly-Poisson manifold and the relations  $pr_j^N \circ \bar{f} = f_j \circ pr_j^M$ , is a poly-Poisson map.

The next example is a particular case of the direct-sum of linear Poisson structures treated in [19, Example 3.9].

**Example 3.5.** Let  $\mathfrak{g}$  be a Lie algebra, and let

$$\mathfrak{g}_{(k)} := \mathfrak{g} \times \cdots \times \mathfrak{g}, \quad \mathfrak{g}_{(k)}^* := \mathfrak{g}^* \times \cdots \times \mathfrak{g}^*.$$

For  $u \in \mathfrak{g}$ , let  $u_j \in \mathfrak{g}_{(k)}$  denote the element  $(0, \dots, 0, u, 0, \dots, 0)$ , with  $u$  in the  $j$ -th entry. Since  $\mathfrak{g}^*$  is equipped with its Lie-Poisson structure,  $\mathfrak{g}_{(k)}^*$  naturally carries a product poly-Poisson structure, as in Example 3.4. More important to us is the following *direct-sum* poly-Poisson structure [19] : over each  $\zeta = (\zeta_1, \dots, \zeta_k) \in \mathfrak{g}_{(k)}^*$ , we define

$$S|_{\zeta} := \{(u_1, \dots, u_k) | u \in \mathfrak{g}\} \subseteq \oplus_{(k)} T_{\zeta}^* \mathfrak{g}_{(k)}^* \cong \oplus_{(k)} \mathfrak{g}_{(k)},$$

and the bundle map  $P : S \rightarrow T\mathfrak{g}_{(k)}^*$ ,

$$P_{\zeta}(u_1, \dots, u_k) := (\text{ad}_u^* \zeta_1, \dots, \text{ad}_u^* \zeta_k) \in T_{\zeta} \mathfrak{g}_{(k)}^* \cong \mathfrak{g}_{(k)}^*.$$

We remark that  $S$  satisfies (ii) in Def. 3.1, not just (3.3).

**3.2. Poly-Poisson structures and poly-symplectic groupoids.** We will now justify our definition of poly-Poisson structure in Def. 3.1 in light of its relation with poly-symplectic groupoids.

Let  $(M, S, P)$  be a  $k$ -poly-Poisson manifold. We saw in Section 3.1 that the vector subbundle  $S \subseteq \oplus_{(k)} T^*M$  is a Lie algebroid, with anchor  $P : S \rightarrow TM$  and bracket (3.1).

**Lemma 3.6.** *Let  $\mu : S \hookrightarrow \oplus_{(k)} T^*M$  be the inclusion. Then  $\mu$  is an IM poly-symplectic form on the Lie algebroid  $S \rightarrow M$ , i.e.,  $\mu$  satisfies (2.6), (2.7) and (2.11).*

*Conversely, any IM poly-symplectic form  $\mu : A \rightarrow \oplus_{(k)} T^*M$  is equivalent to one coming from a  $k$ -poly-Poisson structure.*

*Proof.* Note that (2.6) is just (i) in Def. 3.1, while property (2.7) follows from (iii) in Def. 3.1. Since  $\mu$  is an inclusion,  $\ker(\mu) = \{0\}$ . The second condition in (2.11) is (ii) in Def. 3.1.

On the other hand, given an IM poly-symplectic form  $\mu : A \rightarrow \oplus_{(k)} T^*M$ , we define  $S = \text{Im}(\mu)$ . Note (from the first condition in (2.11)) that  $\mu$  is a vector-bundle isomorphism onto  $S$ , and let  $P : S \rightarrow TM$  be its inverse  $S \rightarrow A$  composed with the anchor  $A \rightarrow TM$ . One may directly verify from conditions (2.6), (2.7) and (2.11) that  $S$  and  $P$  define a  $k$ -poly-Poisson structure, and that  $\mu$  is equivalent to the inclusion  $S \hookrightarrow \oplus_{(k)} T^*M$ .

□

In short, the lemma says that a  $k$ -poly-Poisson manifold  $(M, S, P)$  endows  $S$  with a Lie algebroid structure for which the inclusion  $S \hookrightarrow \oplus_{(k)} T^*M$  is an IM poly-symplectic form, and that any IM poly-symplectic form is equivalent to one of this type.

Following Corollary 2.5, we see that poly-Poisson manifolds are the infinitesimal counterparts of poly-symplectic groupoids, as explained by the next result. For a  $k$ -poly symplectic groupoid  $(\mathcal{G} \rightrightarrows M, \omega)$ , let  $\mu : A \rightarrow \oplus_{(k)} T^*M$  be the bundle map determined by  $\omega$  as in Cor. 2.5. Explicitly, using the natural decomposition  $T\mathcal{G}|_M = TM \oplus A$ ,

$$\mu(u) = \omega^b(u)|_{\oplus_{(k)} TM},$$

for  $u \in A$ .

**Theorem 3.7** (Integration of poly-Poisson structures). *If  $(\mathcal{G} \rightrightarrows M, \omega)$  is a  $k$ -poly-symplectic groupoid, then there exists a unique  $k$ -poly-Poisson structure  $(S, P)$  on  $M$  such that  $S = \text{Im}(\mu)$  while  $P$  is determined by the fact that the target map  $t : \mathcal{G} \rightarrow M$  is a poly-Poisson morphism.*

*Conversely, let  $(M, S, P)$  be a  $k$ -poly-Poisson manifold and  $\mathcal{G} \rightrightarrows M$  be a source-simply-connected groupoid integrating the Lie algebroid  $S \rightarrow M$ . Then there is a  $\omega \in \Omega^2(\mathcal{G}, \mathbb{R}^k)$ , unique up to isomorphism, making  $\mathcal{G}$  into a poly-symplectic groupoid for which  $t : \mathcal{G} \rightarrow M$  is a poly-Poisson morphism.*

We say that a poly-symplectic groupoid *integrates* a poly-Poisson structure if they are related as in the theorem. We observe that this correspondence between source-simply-connected poly-symplectic groupoids and poly-Poisson manifolds (with integrable Lie algebroid) extends the well-known relationship between symplectic groupoids and Poisson manifolds when  $k = 1$ , see [11, 26].

*Proof.* We know from Corollary 2.5 that multiplicative poly-symplectic structures on  $\mathcal{G} \rightrightarrows M$  correspond to IM poly-symplectic forms  $\mu$  on its Lie algebroid  $A \rightarrow M$  via

$$(3.5) \quad i_{u^R} \omega = t^*(\mu(u)),$$

and that  $\mu$  corresponds to a poly-Poisson structure  $(S, P)$  on  $M$ , as described in Lemma 3.6. It remains to verify that condition (3.5) implies that  $t$  is a poly-Poisson map.

Let  $\alpha \in \text{Im}(\mu) = S$ . Then  $\alpha = \mu(u)$  for a unique  $u \in A$ , and  $P(\alpha) = \rho(u)$ . Let  $(S_\omega, P_\omega)$  be the poly-Poisson structure defined by  $\omega$ , as in (3.4). Then (3.5) says that  $t^*\alpha \in S_\omega$  and  $u^R = P_\omega(t^*\alpha)$ ; the fact that on any Lie groupoid we have  $t_*(u^R) = \rho(u)$  implies that  $t_*P_\omega(t^*\alpha) = P(\alpha)$ , i.e.,  $t$  is a poly-Poisson map.  $\square$

**Remark 3.8.** *Given a  $k$ -poly-symplectic groupoid  $(\mathcal{G} \rightrightarrows M, \omega)$ , the uniqueness of the induced poly-Poisson structure  $(S, P)$  on  $M$  follows from Remark 3.5: note that  $S$  is determined by  $\omega$ , while  $P$  is completely defined from the property that  $t$  is a poly-Poisson map.*

We illustrate the correspondence in Theorem 3.7 with some simple examples.

**Example 3.6.** The  $k$ -poly-symplectic manifold  $\oplus_{(k)} T^*Q$  of Example 2.1 is a poly-symplectic groupoid over  $Q$ , with respect to fibrewise addition; the source and target maps coincide with the projection  $\oplus_{(k)} T^*Q \rightarrow Q$ . The corresponding  $k$ -poly-Poisson structure on  $Q$  is the trivial one, given by  $S := \oplus_{(k)} T^*Q$  and  $P = 0$ . Note that Example 3.3 shows other poly-Poisson structures on  $Q$  for which the projection  $\oplus_{(k)} T^*Q \rightarrow Q$  is a poly-Poisson map, but there is only one with the bundle  $S$  prescribed by Theorem 3.7.

**Example 3.7.** Let  $(M, \omega)$  be a  $k$ -poly-symplectic manifold, that we view as a poly-Poisson manifold as in Example 3.2. The non-degeneracy of  $\omega$  implies that the Lie algebroid  $(S_\omega, P_\omega)$  is isomorphic to  $TM$ . Hence this poly-Poisson structure is integrated by the pair groupoid  $M \times M \rightrightarrows M$ , equipped with the  $k$ -poly-symplectic structure  $t^*\omega - s^*\omega \in \Omega^2(M \times M, \mathbb{R}^k)$ , where  $s, t$  are the source and target maps on the pair groupoid, i.e  $t(x, y) = x$  and  $s(x, y) = y$ .

**Example 3.8.** Consider Poisson manifolds  $(M_j, \pi_j)$ ,  $j = 1, \dots, k$ , and equip  $M = M_1 \times \dots \times M_k$  with the product poly-Poisson structure of Example 3.4. For each

$j$ , suppose that  $(\mathcal{G}_j \rightrightarrows M_j, \omega_j)$  is a symplectic groupoid integrating  $(M_j, \pi_j)$ . Then the product poly-symplectic groupoid  $\mathcal{G} = \mathcal{G}_1 \times \dots \times \mathcal{G}_k$  of Prop. 2.2 integrates  $M$ . Indeed, one may verify that the bundle  $S$  on  $M$  described in Example 3.4 agrees with the one prescribed by Theorem 3.7 and, as a consequence of the construction of poly-Poisson maps as products of Poisson maps in Example 3.4, the target map on  $\mathcal{G} \rightrightarrows M$  is a poly-Poisson map.

**Example 3.9** (Lie-Poisson structures). Let  $\mathbb{G}$  be a Lie group and  $\mathfrak{g}$  its Lie algebra. As seen in Example 2.1,  $\oplus_{(k)} T^*\mathbb{G}$  has a natural  $k$ -poly-symplectic structure  $\omega$ .

The diagonal coadjoint action of  $\mathbb{G}$  on  $\mathfrak{g}_{(k)}^*$ , denoted by  $\text{Ad}_g^*$ , endows  $\mathbb{G} \times \mathfrak{g}_{(k)}^*$  with a groupoid structure over  $\mathfrak{g}_{(k)}^*$ , with source and target maps given by

$$s(g, \zeta) = \zeta, \quad t(g, \zeta) = \text{Ad}_g^* \zeta$$

and multiplication  $m((g, \zeta), (h, \eta)) = (gh, \eta)$  if  $\text{Ad}_h^* \eta = \zeta$ . Using the identification  $T^*\mathbb{G} \cong \mathbb{G} \times \mathfrak{g}^*$  (by right translation), we see that

$$\oplus_{(k)} T^*\mathbb{G} \cong \mathbb{G} \times \mathfrak{g}_{(k)}^*,$$

so we may consider  $\oplus_{(k)} T^*\mathbb{G}$  as a Lie groupoid, and its poly-symplectic structure  $\omega$  makes it into a poly-symplectic groupoid. This structure integrates the direct-sum poly-Poisson structure on  $\mathfrak{g}_{(k)}^*$  described in Example 3.5. Indeed,  $t$  has the Poisson maps  $\mathbb{G} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  as its coordinates, so it is a poly-Poisson map. And one can check that the bundle  $S$  of the direct-sum poly-Poisson structure is the one induced by the poly-symplectic structure  $\omega$  according to Theorem 3.7.

**Remark 3.9.** *More generally: following [19] there is a direct-sum poly-Poisson structure on  $A^* \oplus \dots \oplus A^*$  where  $A^* \rightarrow M$  is endowed with the linear Poisson structures (defined on the dual bundle to the Lie algebroid  $A \rightarrow M$ ). Each  $A^*$  is integrated by the symplectic groupoid  $T^*\mathcal{G} \rightrightarrows A^*$ , where  $\mathcal{G} \rightrightarrows M$  is the groupoid integrating  $A$ , and it can be similarly proved that the direct sum  $T^*\mathcal{G} \oplus \dots \oplus T^*\mathcal{G}$  over  $\mathcal{G}$  is the poly-symplectic groupoid integrating  $A^* \oplus \dots \oplus A^*$ .*

**3.3. Poly-symplectic foliation.** It is well known that any Poisson manifold has an underlying symplectic foliation which uniquely determines the Poisson structure. More generally, let  $(S, P)$  be a  $k$ -poly-Poisson structure on  $M$ . Since  $S$  has a Lie algebroid structure, the distribution  $D := P(S) \subseteq TM$  is integrable, and its leaves define a singular foliation on  $M$ . Each leaf  $\iota : \mathcal{O} \hookrightarrow M$  carries an  $\mathbb{R}^k$ -valued 2-form  $\omega_{\mathcal{O}}$  determined by the condition

$$(3.6) \quad \omega_{\mathcal{O}}^\flat : T\mathcal{O} \rightarrow \oplus_{(k)} T^*\mathcal{O}, \quad P(\bar{\eta}) \mapsto \iota^* \bar{\eta}.$$

The fact that the 2-form  $\omega_{\mathcal{O}}$  on  $\mathcal{O}$  is well defined follows from (i) in Def. 3.1, (ii) guarantees that it is non-degenerate and (iii) that it is closed, see [19, Sec. 3]. So  $(S, P)$  determines a singular foliation on  $M$  with  $(k+1)$ -poly-symplectic leaves.

A first remark on the poly-symplectic foliation of a  $k$ -poly-Poisson structure is that, in contrast with the case  $k = 1$ , different  $k$ -poly-Poisson structures may correspond to the same poly-symplectic foliation, as shown in the next example.

**Example 3.10.** Let  $\omega_t$  be a smooth family of  $k$ -poly-symplectic forms on  $M$  parametrized by  $t \in \mathbb{R}$  and define the following vector subbundles of  $\oplus_{(k)} T^*(M \times \mathbb{R})$ :

$$\begin{aligned}
S_1|_{(m,t)} &:= \{(i_X \omega_t, r_1 \oplus \cdots \oplus r_k) | X \in T_m M, r_j \in T_t^* \mathbb{R}\}, \\
S_2|_{(m,t)} &:= \{(i_X \omega_t, r \oplus \cdots \oplus r) | X \in T_m M, r \in T_t^* \mathbb{R}\}, \\
S_3|_{(m,t)} &:= \{(i_X \omega_t, r \oplus 0 \cdots \oplus 0) | X \in T_m M, r \in T_t^* \mathbb{R}\};
\end{aligned}$$

on each  $S_j$  we define  $P_j(i_X \omega_t, \bar{\gamma}) = X$ . Observe that each  $(S_j, P_j)$  is a poly-Poisson structure on  $M \times \mathbb{R}$  but these three  $k$ -poly-Poisson structures have the same poly-symplectic foliation on  $M \times \mathbb{R}$ . Same conclusion holds for the weak-poly-Poisson structure given by

$$S_0|_{(m,t)} := \{(i_X \omega_t, 0) | X \in T_m M\} \quad \text{and} \quad P_0(i_X \omega_t, 0) = X$$

where the poly-symplectic foliation is described on Theorem 3.4 on [19].

We now discuss the possibility of defining a poly-Poisson structure from a poly-symplectic foliation. Given a subspace  $D_m \subseteq T_m M$ , for  $m \in M$ , equipped with a  $(k+1)$ -poly-symplectic form  $\omega_m$ , we consider the subspace  $S_m \subseteq \oplus_{(k)} T^* M$  given by

$$(3.7) \quad S_m := \{\bar{\eta} \in \oplus_{(k)} T_m^* M \mid \exists X \in D_m, \bar{\eta}|_{\oplus_{(k)} D_m} = i_X \omega_m\},$$

which has dimension  $k(n-p) + p$ , where  $p$  is the dimension of  $D_m$ . One may verify that  $S_m^\circ = \{0\}$  and there is a well-defined map  $P_m : S_m \rightarrow D_m \subseteq T_m M$ ,

$$(3.8) \quad P_m(\bar{\eta}) = X, \quad \text{where} \quad \bar{\eta}|_{\oplus_{(k)} D_m} = i_X \omega_m.$$

Given now a *regular* poly-symplectic foliation on  $M$ , letting  $D$  be its tangent distribution, we use the previous pointwise construction to see that (3.7) defines a subbundle  $S \subseteq \oplus_{(k)} T^* M$ , satisfying  $S^\circ = \{0\}$ , and equipped with a bundle map  $P : S \rightarrow TM$ . Moreover, using the fact that the  $\mathbb{R}^k$ -valued form defined on each leaf is closed, it follows that  $(S, P)$  satisfies (iii) in Def. 3.1, so it is a poly-Poisson structure. In conclusion we have the following proposition (see [19, Sec. 3]),

**Proposition 3.10.** *If  $(D, \omega)$  is a regular  $k$ -poly-symplectic foliation on  $M$  then  $(S, P)$ , defined pointwise by (3.7) and (3.8), is a  $k$ -poly-Poisson on  $M$ .*

In particular, if the regular  $k$ -poly-symplectic foliation on  $M$  comes from a weak-poly-Poisson structure as in [19, Theorem 3.4], then the poly-Poisson structure on the proposition is an “extension” of the weak-poly-Poisson structure. In order to illustrate last claim and the poly-Poisson structure from (3.7) and (3.8) we apply the proposition to the regular poly-symplectic foliation given in Example 3.10, which is the same for each poly-Poisson structures  $(S_j, P_j)$  for  $j = 1, 2, 3$  and for the weak-poly-Poisson  $(S_0, P_0)$ , and get the “maximal” poly-Poisson structure  $(S_1, P_1)$ .

**3.4. Relation with AV-Dirac structures.** It is well known that Poisson structures on  $M$  can be understood as special types of Dirac structures in the Courant algebroid  $TM \oplus T^*M$  [12]. As we now see, this picture can be generalized to poly-Poisson structures. We consider the bundle  $\mathbb{A} := TM \oplus (\oplus_{(k)} T^*M)$ , equipped with the  $(\mathbb{R}^k$ -valued) fibrewise inner product

$$\langle X \oplus \bar{\eta}, Y \oplus \bar{\gamma} \rangle := i_X \bar{\gamma} + i_Y \bar{\eta},$$

and bracket on sections of  $\mathbb{A}$  given by

$$[[X \oplus \bar{\eta}, Y \oplus \bar{\gamma}]] := [X, Y] \oplus \mathcal{L}_X \bar{\gamma} - i_Y d\bar{\eta}.$$

For  $k = 1$ , this is the standard Courant algebroid  $TM \oplus T^*M$ . In general, this is a very particular case of the *AV-Courant algebroids* introduced in [24, Sec. 2] (with respect to the Lie algebroid  $A = TM$  and representation on  $V = M \times \mathbb{R}^k \rightarrow M$  given by the Lie derivative  $\mathcal{L}_X(f_1, \dots, f_k) = (\mathcal{L}_X f_1, \dots, \mathcal{L}_X f_k)$  on  $C^\infty(M, \mathbb{R}^k)$ ).

Following [24], one may consider *AV-Dirac structures* on any AV-Courant algebroid: these are subbundles  $L \subseteq \mathbb{A}$  which are *lagrangian*, i.e.,

$$(3.9) \quad L = L^\perp,$$

with respect to the fibrewise inner product, and which are involutive with respect to the bracket  $[\![\cdot, \cdot]\!]$  on  $\Gamma(\mathbb{A})$ . Recall that  $L$  is called *isotropic* if  $L \subset L^\perp$ .

**Example 3.11.** Any  $k$ -poly-symplectic structure  $\omega$  on  $M$  may be seen as an AV-Dirac structure in  $\mathbb{A} := TM \oplus (\oplus_{(k)} T^*M)$  via

$$L := \text{graph}(\omega) = \{X \oplus i_X \omega \mid X \in TM\}.$$

Note that this  $L$  satisfies the additional condition

$$(3.10) \quad L \cap (\oplus_{(k)} T^*M) = \{0\}.$$

In fact, poly-symplectic structures on  $M$  are in one-to-one correspondence with AV-Dirac structures which project isomorphically over  $TM$  and satisfy  $L \cap TM = \{0\}$  and (3.10).

Our goal now is to define, in the same way, a subbundle  $L$  from a poly-Poisson structure  $(S, P)$ , i.e. consider

$$L = \{P(\bar{\eta}) \oplus \bar{\eta} \mid \bar{\eta} \in S\}.$$

Note that  $L$  is isotropic as a consequence of (i) in Definition 3.1. But, as we now see, the lagrangian condition generally fails.

**Example 3.12.** Let  $\mathfrak{g}$  be a Lie algebra and consider the poly-Poisson structure on  $\mathfrak{g}_{(2)}^*$  as in Example 3.5. Observe that  $L$  over the point  $\zeta = (0, 0) \in \mathfrak{g}_{(2)}^*$  can be written as

$$L_\zeta = \{(0, 0) \oplus ((u, 0), (0, u)) \mid u \in \mathfrak{g}\}.$$

But for any  $v_1, v_2, w_1, w_2 \in \mathfrak{g}$  we have  $(0, 0) \oplus ((v_1, v_2), (w_1, w_2)) \in L^\perp$ , hence  $L$  is properly contained in  $L^\perp$ .

Therefore, in general, poly-Poisson structures are not AV-Dirac structures. In order to include poly-Poisson structures in the formalism of AV-Courant algebroids, one then needs to relax the lagrangian condition (3.9).

Let us consider subbundles  $L \subseteq TM \oplus (\oplus_{(k)} T^*M)$  satisfying

$$(3.11) \quad L = L^\perp \cap (L + TM).$$

Note that (3.9) implies that (3.11) holds, but the converse is not true.

The following results characterize  $k$ -poly-Poisson structures as subbundles of  $\mathbb{A} = TM \oplus (\oplus_{(k)} T^*M)$ :

**Proposition 3.11.** *There is a one-to-one correspondence among the following:*

- (a)  $k$ -poly-Poisson structures  $(S, P)$  on  $M$ ,
- (b) Involutive, isotropic subbundles  $L \subset \mathbb{A}$  satisfying  $L^\perp \cap TM = \{0\}$ ,
- (c) Involutive subbundles  $L \subset \mathbb{A}$  satisfying (3.11) and  $L \cap TM = \{0\}$ .

*Proof.* Given a  $k$ -poly-Poisson structure  $(S, P)$ , we define the subbundle  $L \subset \mathbb{A}$  by

$$(3.12) \quad L = \{P(\bar{\eta}) \oplus \bar{\eta} \mid \bar{\eta} \in S\}.$$

This bundle is isotropic by condition (i) in Def. 3.1, condition (ii) amounts to  $L^\perp \cap TM = \{0\}$  while (iii) is equivalent to the involutivity of  $L$ . Conversely, given  $L$  as in (b), the image of the natural projection  $L \rightarrow \oplus_{(k)} T^*M$  defines a vector bundle  $S$  and a bundle map  $P : S \rightarrow TM$  by

$$P(\bar{\eta}) = X \text{ if and only if } X \oplus \bar{\eta} \in L,$$

in such a way that  $(S, P)$  is a  $k$ -poly-Poisson structure. This gives the correspondence between (a) and (b).

For a  $k$ -poly-Poisson structure  $(S, P)$  and  $L$  as in (3.12), one may directly verify that (i) in Def. 3.1 implies that (3.11) holds, while (ii) implies that  $L \cap TM = \{0\}$ , so  $L$  satisfies the properties in (c). It remains to check that given an  $L$  as in (c), then it satisfies the properties described in (b). Note that (3.11) implies that  $L$  is isotropic and that  $L \cap TM = L^\perp \cap TM$ , so that  $L^\perp \cap TM = \{0\}$ .  $\square$

**Remark 3.12.** For  $k = 1$ , the objects in (b) and (c) are just usual Dirac structures on  $M$ , satisfying the additional condition  $L \cap TM = \{0\}$  (conditions (3.9) and (3.11) turn out to be equivalent for  $k = 1$ ), while the objects in (a) are usual Poisson structures. So for  $k = 1$  Prop. 3.11 boils down to the known characterization of Poisson structures as particular types of Dirac structures.

#### 4. SYMMETRIES AND REDUCTION

We now discuss poly-Poisson structures and poly-symplectic groupoids in the presence of symmetries, with the aim of using reduction as a tool for integration of poly-Poisson manifolds, along the lines of [30, 17].

**4.1. Poly-Poisson actions.** An action  $\varphi$  of a Lie group  $\mathbb{G}$  on a  $k$ -poly-Poisson manifold  $(M, S, P)$  is a *poly-Poisson action* if for each  $g \in \mathbb{G}$  the diffeomorphism  $\varphi_g : M \rightarrow M$  is a poly-Poisson morphism (Def. 3.4). In the case of  $k$ -poly-symplectic manifold  $(M, \omega)$ , this means that  $\varphi_g^* \omega = \omega$ , see Example 3.2.

Let us consider a poly-Poisson action  $\varphi$  of a Lie group  $\mathbb{G}$  on  $(M, S, P)$ , and let us assume henceforth that this action is free and proper, so that we have a principal  $\mathbb{G}$ -bundle:

$$(4.1) \quad \Pi : M \rightarrow M/\mathbb{G}.$$

Let  $V \subseteq TM$  denote the vertical bundle defined by this action.

It is well-known that, when  $k = 1$ , i.e.,  $M$  is an ordinary Poisson manifold,  $M/\mathbb{G}$  inherits a Poisson structure for which  $\Pi$  is a Poisson map. For poly-Poisson manifolds, we will need additional conditions. We call the action  $\varphi$  *reducible* if

$$(4.2) \quad \begin{cases} (a) \ S \cap \oplus_k \text{Ann}(V) \text{ has constant rank,} \\ (b) \ (S \cap \oplus_k \text{Ann}(V))^\circ \subset V. \end{cases}$$

The projection map (4.1) induces a map  $d\Pi_{(k)} : \oplus_{(k)} TM \rightarrow \Pi^*(\oplus_{(k)} T(M/\mathbb{G}))$ , and its transpose is an injective bundle map  $\Pi^*(\oplus_{(k)} T^*(M/\mathbb{G})) \rightarrow \oplus_{(k)} T^*M$ , whose image is the subbundle  $\oplus_{(k)} \text{Ann}(V) \subseteq \oplus_{(k)} T^*M$ . So we have an induced isomorphism

$$(4.3) \quad \Pi^*(\oplus_{(k)} T^*(M/\mathbb{G})) \xrightarrow{\sim} \oplus_{(k)} \text{Ann}(V).$$

The next result is analogous to [19, Thm. 4.1] (but stated for our stronger notion of poly-Poisson structure).

**Theorem 4.1.** *Let us consider a poly-Poisson  $\mathbb{G}$ -action on a  $k$ -poly-Poisson manifold  $(M, S, P)$  which is free and proper, and reducible. Then  $M/\mathbb{G}$  inherits a  $k$ -poly-Poisson structure  $(S_{red}, P_{red})$ , where the subbundle  $S_{red} \subseteq \oplus_{(k)} T^*(M/\mathbb{G})$  corresponds to  $S \cap \oplus_{(k)} \text{Ann}(V)$  via (4.3), and  $P_{red}$  is unique so that the quotient map (4.1) is a  $k$ -poly-Poisson morphism.*

*Proof.* The first condition in (4.2) guarantees that  $S_{red} \subseteq \oplus_{(k)} T^*(M/\mathbb{G})$ , defined by the condition that  $\Pi^*S_{red}$  is isomorphic to  $S \cap \oplus_{(k)} \text{Ann}(V)$  under (4.3), is a vector subbundle. Note that we have a natural map  $\Pi^*(S_{red}) \rightarrow \Pi^*(T(M/\mathbb{G}))$  given by the composition

$$(4.4) \quad \Pi^*(S_{red}) \xrightarrow{d\Pi_{(k)}^*} S \cap \oplus_{(k)} \text{Ann}(V) \xrightarrow{P} TM \xrightarrow{d\Pi} \Pi^*(T(M/\mathbb{G})),$$

and this defines a bundle map

$$(4.5) \quad P_{red} : S_{red} \rightarrow T(M/\mathbb{G})$$

as a consequence of the  $\mathbb{G}$ -invariance of  $(S, P)$ .

To check that  $(S_{red}, P_{red})$  defines a  $k$ -poly-Poisson structure on  $M/\mathbb{G}$ , one must verify that it satisfies conditions (i), (ii), and (iii) in Def. 3.1. Condition (i) follows directly from the definition of  $(S_{red}, P_{red})$  and the fact that this condition is satisfied by  $(S, P)$ . It is also routine to check that condition (iii) holds for  $(S_{red}, P_{red})$ , given that it holds for  $(S, P)$ .

As for condition (ii), it is a consequence of property (b) in (4.2). Indeed, by the way  $S_{red}$  is defined, the fact that  $\bar{X} \in S_{red}^\circ$  implies that  $\bar{X} = d\Pi(X)$ , for  $X \in (S \cap \oplus_{(k)} \text{Ann}(V))^\circ$ . But then (b) in (4.2) implies that  $\bar{X} = d\Pi(X) = 0$ .

It is also clear from the definition of  $P_{red}$  that  $\Pi$  is a poly-Poisson map. □

We mention two concrete examples, discussed in [19].

**Example 4.1.**

- (a) Let  $Q$  be a manifold equipped with a free and proper  $\mathbb{G}$ -action, and let  $(M = \oplus_{(k)} T^*Q, \omega)$  be the poly-symplectic manifold of Example 2.1. We keep the notation  $\text{pr}_j : M \rightarrow T^*Q$  for the natural projection onto the  $j$ th-factor. The cotangent lift of the  $\mathbb{G}$ -action on  $Q$  defines an action on  $T^*Q$ , which induces a  $\mathbb{G}$ -action on  $M$  which preserves the poly-symplectic structure (i.e., it is a poly-Poisson action), and there is a natural identification

$$M/\mathbb{G} \cong \oplus_{(k)} (T^*Q/\mathbb{G}).$$

We observe here that both conditions in (4.2) hold, i.e., the  $\mathbb{G}$ -action on  $M$  is reducible. To verify this fact, let  $V \subseteq TM$  be the vertical bundle of the  $\mathbb{G}$ -action on  $M$ , so that  $V_j = d\text{pr}_j(V) \subseteq T(T^*Q)$  is the vertical bundle of the



$\mathbb{G}$ -action on the  $j$ th-factor  $T^*Q$ . Note that the natural projection  $T^*Q \rightarrow Q$  induces a projection of  $V_j^{\omega_{can}}$  onto  $TQ$ , and one then sees that

$$V_1^{\omega_{can}} \times_{TQ} \dots \times_{TQ} V_k^{\omega_{can}} \subseteq T(T^*Q) \times_{TQ} \dots \times_{TQ} T(T^*Q) = TM$$

is a vector subbundle, that we denote by  $W$ . One can now check that

$$(4.6) \quad S_\omega \cap \oplus_{(k)} \text{Ann}(V) = \{i_X \omega \mid X \in W\},$$

from where one concludes that condition (a) of (4.2) holds. From (4.6), one directly sees that

$$\begin{aligned} (S_\omega \cap \oplus_{(k)} \text{Ann}(V))^\circ &= (V_1^{\omega_{can}})^{\omega_{can}} \times_{TQ} \dots \times_{TQ} (V_k^{\omega_{can}})^{\omega_{can}} \\ &= V_1 \times_{TQ} \dots \times_{TQ} V_k = V, \end{aligned}$$

showing that (b) of (4.2) also holds. So the action is reducible. As shown in [19, Ex. 4.3], the reduced poly-Poisson structure on  $\oplus_{(k)}(T^*Q/\mathbb{G})$  is the one defined by direct-sum of the natural linear Poisson structure on  $T^*Q/\mathbb{G}$  (dual to the Atiyah algebroid  $TQ/\mathbb{G}$  of the principal bundle  $Q \rightarrow Q/\mathbb{G}$ ).

- (b) In the particular case of  $Q = \mathbb{G}$  with the action by left multiplication, as shown in [19, Ex. 4.2], the poly-Poisson reduction of  $\oplus_{(k)} T^*\mathbb{G}$  with respect to the lifted  $\mathbb{G}$ -action is identified with  $\mathfrak{g}_{(k)}^*$  of Example 3.5.

**4.2. Hamiltonian actions on poly-symplectic manifolds.** We now consider poly-Poisson actions on poly-symplectic manifolds in the presence of moment maps.

Let  $(M, \omega)$  be a  $k$ -poly-symplectic manifold equipped with a poly-Poisson action of  $\mathbb{G}$ , denoted by  $\varphi$ . Consider the diagonal coadjoint action of  $\mathbb{G}$  on the space  $\mathfrak{g}_{(k)}^*$ . This action is called *hamiltonian* [18, 27] if there is a *moment map*, i.e., a map  $J : M \rightarrow \mathfrak{g}_{(k)}^*$  that satisfies

$$(4.7) \quad \text{(i) } J \circ \varphi_g = \text{Ad}_g^* \circ J \quad \text{and} \quad \text{(ii) } i_{u_M} \omega = d\langle J, u \rangle.$$

for all  $u \in \mathfrak{g}$ . Here  $u_M \in \mathfrak{X}(M)$  denotes the infinitesimal generator corresponding to  $u \in \mathfrak{g}$ .

**Example 4.2.** Let  $(M, \omega)$  be a poly-symplectic manifold such that  $\omega = -d\theta$ , and assume that  $\mathbb{G}$  acts on  $M$  preserving the 1-form  $\theta$ . Then the maps  $J_1, \dots, J_k : M \rightarrow \mathfrak{g}^*$  defined by  $\langle J_l, u \rangle = \theta_l(u_M)$ ,  $u \in \mathfrak{g}$ , define a moment map for the action.

A particular case of this example is when  $M = \oplus_{(k)} T^*Q$  (as in Example 2.1) and the action of  $\mathbb{G}$  on  $M$  is the lift of an action on  $Q$ , see Example 4.1(a). Here the moment map is  $\langle J(\bar{\eta}), u \rangle = (\langle \eta_j, u_Q \rangle)_{j=1, \dots, k}$ .

The following observation generalizes a well-known fact in Poisson geometry. Consider  $\mathfrak{g}_{(k)}^*$  with the poly-Poisson structure of Example 3.5.

**Proposition 4.2.** *The moment map  $J : M \rightarrow \mathfrak{g}_{(k)}^*$  of a hamiltonian action of  $\mathbb{G}$  on  $(M, \omega)$  is a poly-Poisson morphism.*

*Proof.* Denote the poly-Poisson structure on  $\mathfrak{g}_{(k)}^*$  by  $(S, P)$ , as in Example 3.5. Consider  $(u_1, \dots, u_k) \in S|_\zeta$ , and  $Y \in T_x M$  with  $J(x) = \zeta$ . By condition (ii) in (4.7) we have

$$\begin{aligned} (J^*(u_1, \dots, u_k))(Y) &= (u_1, \dots, u_k)(dJ(Y)) = (\langle dJ(Y), u_j \rangle)_{j=1, \dots, k} \\ &= (\langle dJ_j(Y), u \rangle)_{j=1, \dots, k} = (i_{u_M} \omega)|_x(Y), \end{aligned}$$

hence  $(J^*(u_1, \dots, u_k)) = i_{u_M} \omega|_x \in \text{Im}(\omega^\flat)|_x$ .

Recall the bundle maps of the poly-Poisson structures on  $M$  and  $\mathfrak{g}_{(k)}^*$ :

$$P_\omega(J^*(u_1, \dots, u_k))|_x = P_\omega(i_{u_M} \omega|_x) = u_M|_x,$$

$$P|_\zeta(u_1, \dots, u_k) = (\text{ad}_u^* \zeta_j)_{j=1, \dots, k} = (u_{\mathfrak{g}^*}(\zeta_j))_{j=1, \dots, k}.$$

From condition (i) in (4.7) we can derive that  $dJ_j(u_M(x)) = u_{\mathfrak{g}^*}(\zeta_j)$ , therefore on points  $\zeta = J(x)$  we obtain

$$dJ(P_\omega|_x(J^*(u_1, \dots, u_k))) = (dJ_j(u_M(x))) = (u_{\mathfrak{g}^*}(\zeta_j)) = P|_\zeta(u_1, \dots, u_k).$$

□

Let us consider a Hamiltonian  $\mathbb{G}$ -action on a  $k$ -poly-symplectic manifold  $(M, \omega)$ , with moment map  $J : M \rightarrow \mathfrak{g}_{(k)}^*$ . Let  $\zeta \in \mathfrak{g}_{(k)}^*$  be a *clean value* for  $J$ , i.e.,

$$(4.8) \quad \begin{cases} J^{-1}(\zeta) \text{ is a submanifold of } M, \\ \ker(d_x J) = T_x J^{-1}(\zeta), \text{ for all } x \in J^{-1}(0). \end{cases}$$

The submanifold  $J^{-1}(\zeta)$  is invariant by the action of  $\mathbb{G}_\zeta$ , the isotropy group of  $\zeta$  with respect to the diagonal coadjoint action. We assume that the  $\mathbb{G}_\zeta$ -action on  $J^{-1}(\zeta)$  is free and proper, so we can consider the reduced manifold

$$M_\zeta := J^{-1}(\zeta)/\mathbb{G}_\zeta.$$

We let  $\Pi_\zeta : J^{-1}(\zeta) \rightarrow M_\zeta$  be the natural projection map, and  $i_\zeta : J^{-1}(\zeta) \rightarrow M$  the inclusion. We denote by  $V_\zeta \subseteq TJ^{-1}(\zeta)$  the vertical bundle with respect to the  $\mathbb{G}_\zeta$ -action. It follows from (i) in (4.7) that  $V_\zeta = V \cap TJ^{-1}(\zeta)$ , while (ii) implies that

$$(4.9) \quad V_\zeta \subseteq \ker(i_\zeta^* \omega).$$

This last condition, together with the  $\mathbb{G}_\zeta$ -invariance of  $i_\zeta^* \omega$ , implies that  $i_\zeta^* \omega$  is basic, i.e., there exists a (unique) closed form  $\omega_{red} \in \Omega^2(M_\zeta, \mathbb{R}^k)$  so that

$$(4.10) \quad \Pi_\zeta^* \omega_{red} = i_\zeta^* \omega.$$

In general, however, the form  $\omega_{red}$  fails to be poly-symplectic, as it may be degenerate; indeed, it is nondegenerate if and only if we have an equality in (4.9).

Note that (ii) in (4.7) says that

$$TJ^{-1}(\zeta) = \ker(dJ) = V^\omega,$$

where, for a subbundle  $W \subseteq TM$ , we use the notation  $W^\omega = \{Y \in TM, | \omega(X, Y) = 0 \ \forall X \in W\}$ . Writing  $S = \text{Im}(\omega^\flat)$ , one may also check that

$$(\ker(dJ))^\omega = (V^\omega)^\omega = (S \cap \oplus_k \text{Ann}(V))^\circ,$$

from where we conclude that

$$\ker(i_\zeta^* \omega) = (TJ^{-1}(\zeta))^\omega \cap TJ^{-1}(\zeta) = (S \cap \oplus_k \text{Ann}(V))^\circ \cap TJ^{-1}(\zeta).$$

Comparing with (4.9), we conclude the following:

**Proposition 4.3.** *The reduced form  $\omega_{red} \in \Omega^2(M_\zeta, \mathbb{R}^k)$  defined by (4.10) is poly-symplectic if and only if*

$$(4.11) \quad (S \cap \oplus_k \text{Ann}(V))^\circ \cap TJ^{-1}(\zeta) \subseteq V_\zeta = V \cap TJ^{-1}(\zeta).$$

A similar, but not equivalent, result of the previous condition was stated on [27, Lemma 3.16].

**Example 4.3.** Consider symplectic manifolds  $(M_j, \omega_j)_{j=1, \dots, k}$  each of them carrying a Hamiltonian action of a Lie group  $\mathbb{G}_j$  with respective moment map  $J_j : M_j \rightarrow \mathfrak{g}_j^*$ . On the product  $k$ -poly-symplectic manifold  $(M, \omega)$  (see Section 2.1) there is a poly-symplectic hamiltonian action given by the product action of  $\mathbb{G} := \prod_{j=1}^k \mathbb{G}_j$  on  $M$  and the moment map  $J : M \rightarrow \oplus_{(k)}(\prod_{j=1}^k \mathfrak{g}_j^*)$ ,  $J(m) = \oplus_{j=1}^k(0, \dots, 0, J_j(m_j), 0, \dots, 0)$ .

Let  $\zeta = \oplus_{j=1}^k(0, \dots, 0, \zeta_j, 0, \dots, 0) \in \oplus_{(k)}(\prod_{j=1}^k \mathfrak{g}_j^*)$  where  $\zeta_j \in \mathfrak{g}_j^*$  is a clean value for  $J_j$ . Then  $J^{-1}(\zeta) = \prod_{j=1}^k J_j^{-1}(\zeta_j)$  and, assuming that each  $\mathbb{G}_{\zeta_j}$  acts freely and properly on  $J_j^{-1}(\zeta_j)$ , then

$$M_\zeta := J^{-1}(\zeta)/\mathbb{G}_\zeta = \prod_{j=1}^k J_j^{-1}(\zeta_j)/\mathbb{G}_{\zeta_j} = \prod_{j=1}^k M_{j, \zeta_j},$$

and the reduced  $\mathbb{R}^k$ -valued 2-form on  $M_\zeta$  is the product  $k$ -poly-symplectic form defined by the reduced symplectic forms on  $M_{j, \zeta_j}$ .

The moment-map reduction of Prop. 4.3 can now be compared with the quotient of poly-Poisson structures in Theorem 4.1.

Assuming that the  $\mathbb{G}$ -action on  $M$  is free and proper, and that  $\zeta$  is a clean value of a moment map  $J : M \rightarrow \mathfrak{g}_{(k)}^*$ , it follows that the  $\mathbb{G}_\zeta$ -action on  $J^{-1}(\zeta)$  is also free and proper, and we have the following diagram of submersions and natural inclusions:

$$(4.12) \quad \begin{array}{ccc} J^{-1}(\zeta) & \xrightarrow{i_\zeta} & M \\ \Pi_\zeta \downarrow & & \downarrow \Pi \\ M_\zeta & \longrightarrow & M/\mathbb{G} \end{array}$$

**Proposition 4.4.** *Let  $(M, \omega)$  be a poly-symplectic manifold equipped with a hamiltonian  $\mathbb{G}$ -action with moment map  $J : M \rightarrow \mathfrak{g}_{(k)}^*$ . Assume that the  $\mathbb{G}$ -action on  $M$  is free, proper and reducible (4.2). If  $\zeta \in \mathfrak{g}_{(k)}^*$  is a clean value for the moment map, then:*

- (a) *The reduced manifold  $M_\zeta = J^{-1}(\zeta)/\mathbb{G}_\zeta$  carries a natural poly-symplectic form defined by equation (4.10);*
- (b) *The poly-symplectic manifold  $M_\zeta$  sits in  $M/\mathbb{G}$  as a union of poly-symplectic leaves of the reduced poly-Poisson manifold on  $M/\mathbb{G}$  (given by Thm. 4.1).*

*Proof.* Note that (4.2)(b) directly implies (4.11), so the reduced form  $\omega_{red}$  on  $M_\zeta$  is indeed poly-symplectic, proving part (a).

By the moment-map condition 4.7(ii),  $X \in \ker(dJ)$  if and only if  $(i_X \omega)(u_M) = 0$  for all  $u \in \mathfrak{g}$ , therefore

$$(4.13) \quad \begin{aligned} TJ^{-1}(\zeta) &= \{X \in TM \mid i_X \omega \in \oplus_k \text{Ann}(V)\} \\ &= P_\omega(S_\omega \cap \oplus_k \text{Ann}(V)) = P_\omega(d\Pi_{(k)}^* S_{red}). \end{aligned}$$

It follows from (4.12) and the construction of the reduced poly-Poisson structure, see (4.4) and (4.5), that

$$TM_\zeta = d\Pi_\zeta(TJ^{-1}(\zeta)) = d\Pi(P_\omega(d\Pi_{(k)}^*S_{red})) = P_{red}(S_{red}).$$

Hence  $M_\zeta$  is a union of poly-symplectic leaves in  $M/\mathbb{G}$ . It remains to check that the poly-symplectic structures (the one coming from reduction and the one induced from the poly-Poisson structure on  $M/\mathbb{G}$ ) agree.

Consider  $\bar{X} = d\Pi_\zeta(X)$ ,  $\bar{Y} = d\Pi_\zeta(Y) \in TM_\zeta$ , with  $X, Y$  tangent to  $J^{-1}(\zeta)$ , and let us compute the two 2-forms on them. For the leafwise poly-symplectic form  $\omega_L$ , we have (see (3.6))

$$\omega_L(\bar{X}, \bar{Y}) = \bar{\eta}_r(\bar{Y}) = (\Pi_\zeta^*\omega_L)(X, Y),$$

where  $\bar{\eta}_r$  is such that  $\bar{X} = P_{red}(\bar{\eta}_r)$ . Letting  $\bar{\eta} = d\Pi_{(k)}^*(\bar{\eta}_r) \in S_\omega \cap \oplus_k \text{Ann}(V)$ , then

$$(\Pi_\zeta^*\omega_L)(X, Y) = \bar{\eta}_r(d\Pi_\zeta(Y)) = \bar{\eta}(Y).$$

Note that there exists a unique  $X_0 \in TM$  such that  $\bar{\eta} = i_{X_0}\omega \in \oplus_k \text{Ann}(V)$ . By (4.13), we know that  $X_0 \in TJ^{-1}(\zeta)$ . Furthermore,

$$d\Pi_\zeta(X_0) = d\Pi(P_\omega(\bar{\eta})) = d\Pi(P_\omega(d\Pi_{(k)}^*(\bar{\eta}_r))) = P_{red}(\bar{\eta}_r) = \bar{X},$$

so  $d\Pi_\zeta(X_0) = d\Pi_\zeta(X)$ . Recalling that  $\Pi_\zeta^*\omega_{red} = i_\zeta^*\omega$ , we see that

$$(\Pi_\zeta^*\omega_{red})(X, Y) = (\Pi_\zeta^*\omega_{red})(X_0, Y) = (i_{X_0}\omega)(Y) = \bar{\eta}(Y) = (\Pi_\zeta^*\omega_L)(X, Y),$$

showing that  $\omega_{red} = \omega_L$  on  $M_\zeta$ .  $\square$

**Example 4.4.**

- (a) Let us consider a  $\mathbb{G}$ -action on  $Q$  and its lift to  $M = \oplus_{(k)} T^*Q$  as in Example 4.1(a). The action on  $M$  is hamiltonian, and using the explicit formula for the moment map in Example 4.2 one sees that its poly-symplectic reduction at  $\zeta = 0$  is  $\oplus_{(k)} T^*(Q/\mathbb{G})$ , with the poly-symplectic form of Example 2.1. Proposition 4.4(b) realizes  $\oplus_{(k)} T^*(Q/\mathbb{G})$  as a poly-symplectic leaf of  $M/\mathbb{G}$ .
- (b) Following Example 4.1(b), in the particular case  $Q = \mathbb{G}$  Proposition 4.4(b) implies that the poly-symplectic reduction of the lifted action on  $\oplus_{(k)} T^*\mathbb{G}$  at level  $\zeta$  (see [27, Sec. 3.3.2]) is identified with the poly-symplectic leaf of  $\mathfrak{g}_{(k)}^*$  through  $\zeta$ , which is the orbit of  $\zeta$  under the diagonal coadjoint action of  $\mathbb{G}$  on  $\mathfrak{g}_{(k)}^*$  (c.f. Example 3.9) equipped with a poly-symplectic generalization of the usual KKS symplectic form on coadjoint orbits, see [19, Example 2.9] and [27, App. A.3].

**4.3. Reduction and integration.** In this section, we show (along the lines of [7, 17]) how passing from poly-Poisson manifolds to poly-symplectic groupoids has the effect of turning poly-Poisson actions into hamiltonian actions, and how poly-symplectic reduction can be used in the construction of poly-symplectic groupoids associated with poly-Poisson quotients.

In the remainder of this section, we will consider the following set-up:

1. A  $k$ -poly-Poisson manifold  $(M, S, P)$ , so that its underlying Lie algebroid is integrable, and  $(\mathcal{G} \rightrightarrows M, \omega)$  the source-simply connected  $k$ -poly-symplectic groupoid integrating it.
2. A poly-Poisson action  $\varphi$  of the Lie group  $\mathbb{G}$  on  $(M, S, P)$ .

Since  $\varphi$  preserves the poly-Poisson structure on  $M$ , the cotangent lift of  $\varphi$  induces an action  $\hat{\varphi} : \mathbb{G} \times S \rightarrow S$  by Lie-algebroid automorphisms, which can be integrated to a poly-symplectic  $\mathbb{G}$ -action on  $\mathcal{G}$ , denoted by

$$\Phi : \mathbb{G} \times \mathcal{G} \rightarrow \mathcal{G}.$$

We will now see that this action on  $\mathcal{G}$  admits a natural moment map (as in (4.7)), so it is hamiltonian.

Let us start by recalling that any action on  $M$  induces a Hamiltonian  $\mathbb{G}$ -action on the symplectic manifold  $T^*M$  with moment map  $J_{can} : T^*M \rightarrow \mathfrak{g}^*$  given by

$$\langle J_{can}(\alpha), u \rangle = \langle \alpha, u_M \rangle$$

for all  $\alpha \in T^*M$  and  $u$  in the Lie algebra  $\mathfrak{g}$  of  $\mathbb{G}$ . We have an induced map  $\oplus_{(k)} T^*M \rightarrow \mathfrak{g}_{(k)}^*$ , that we restrict to  $S$  to define

$$(4.14) \quad J^s : S \rightarrow \mathfrak{g}_{(k)}^*.$$

It is clear from the  $\mathbb{G}$ -equivariance of  $J_{can}$  that  $J^s$  is  $\mathbb{G}$ -equivariant (with respect to the diagonal coadjoint action on  $\mathfrak{g}_{(k)}^*$ ).

The same proof as in [7, Lemma 3.1] shows that, viewing  $\mathfrak{g}_{(k)}^*$  as a trivial Lie algebra,  $J^s$  is a Lie-algebroid morphism. According to our sign conventions, it is more convenient to consider  $-J^s$ , which is also a Lie-algebroid morphism, and integrate it to a Lie-groupoid morphism

$$(4.15) \quad J : \mathcal{G} \rightarrow \mathfrak{g}_{(k)}^*.$$

Just as in [7, Prop. 3.2] one can verify that  $J$  is  $\mathbb{G}$ -equivariant and satisfies:

$$i_{u_{\mathcal{G}}} \omega = d\langle J, u \rangle,$$

for all  $u \in \mathfrak{g}$ , where  $u_{\mathcal{G}}$  is the infinitesimal generator for the action on  $\mathcal{G}$ . In other words,  $J$  is a moment map for the action  $\Phi$  on  $\mathcal{G}$ . The next result summarizes the discussion:

**Proposition 4.5.** *The  $\mathbb{G}$ -action  $\Phi$  on the poly-symplectic groupoid  $(\mathcal{G}, \omega)$  is Hamiltonian with moment map (4.15).*

We now discuss the connection between integration and reduction. We assume from now on that the  $\mathbb{G}$ -action  $\varphi$  on  $M$  is free, proper and reducible (4.2). Then the action  $\Phi$  on  $\mathcal{G}$  is also free and proper [17, Prop. 4.4]. Let  $(S_{red}, P_{red})$  be the quotient poly-Poisson structure on  $M/G$ .

**Theorem 4.6.** *Let  $0 \in \mathfrak{g}_{(k)}^*$  be a clean value for the moment map (4.15). Then  $\mathcal{G}_{red} = J^{-1}(0)/\mathbb{G}$  is a Lie groupoid over  $M/\mathbb{G}$ , and the reduced form  $\omega_{red} \in \Omega^2(\mathcal{G}_{red}, \mathbb{R}^k)$  makes it into a poly-symplectic groupoid integrating  $(S_{red}, P_{red})$ .*

*Proof.* Let  $V_M \subset TM$  be the vertical bundle with respect to the action on  $M$ . According with [7, Lemma 3.1] and condition (a) on (4.2) we conclude that  $(J^s)^{-1}(0) = S \cap \oplus_{(k)} \text{Ann}(V_M)$  is Lie subalgebroid of  $S$ . The  $\mathbb{G}$ -invariance allows us to construct, as in [7, Prop. 4.3], the reduced Lie algebroid  $S_{red} = (J^s)^{-1}(0)/\mathbb{G}$  over  $M/\mathbb{G}$ . Furthermore, the reduced Lie algebroid  $S_{red}$  coincides the one defined by the reduced poly-Poisson structure of Theorem 4.1.

If  $0$  is a clean value for  $J$ ,  $J^{-1}(0)$  is Lie subgroupoid (see [7, Lemma 5.1]). Following the same lines of [7, Prop. 5.2], we see that  $\mathcal{G}_{red} = J^{-1}(0)/\mathbb{G}$  is a Lie groupoid over

$M/\mathbb{G}$ , whose Lie algebroid is  $(S_{red}, P_{red})$ , and the quotient map  $\Pi_0 : J^{-1}(0) \rightarrow \mathcal{G}_{red}$  is a groupoid morphism.

Let  $\omega_{red}$  be the reduced form on  $\mathcal{G}_{red}$ , characterized by  $\Pi_0^* \omega_{red} = i_0^* \omega$ , where  $i_0$  is the natural inclusion of  $J^{-1}(0)$  on  $\mathcal{G}$ . The second part of [7, Prop. 5.2] allows us to conclude that  $\omega_{red}$  is multiplicative.

The fact that the quotient map  $\Pi_0$  and the inclusion  $i_0$  are groupoid morphism yields

$$\Pi_0^*(i_{\bar{u}^R} \omega_{red}) = i_{u^R} \Pi_0^* \omega_{red} = i_0^*(i_{u^R} \omega)$$

for any  $\bar{u} \in S_{red}$  and  $u = d\Pi_{(k)}^* \bar{u} \in S \cap \oplus_{(k)} \text{Ann}(V_M)$ , where  $\bar{u}^R$  and  $u^R$  are the respective right-invariant vector fields on the correspondent Lie groupoid. Moreover, if  $t, t_0, t_{red}$  denote the target maps on the Lie groupoids  $\mathcal{G}$ ,  $J^{-1}(0)$  and  $\mathcal{G}_{red}$ , respectively, we have

$$i_0^*(i_{u^R} \omega) = i_0^*(t^* u) = t_0^* d\Pi_{(k)}^* \bar{u} = \Pi_0^* t_{red}^* \bar{u},$$

which implies that  $i_{\bar{u}^R} \omega_{red} = t_{red}^* \bar{u}$ . It follows from Prop. 2.4 that  $\omega_{red}$  is nondegenerate, so  $(\mathcal{G}_{red}, \omega_{red})$  is a poly-symplectic groupoid, and it integrates  $(S_{red}, P_{red})$ .  $\square$

Theorem 4.6 is a generalization of the following example.

**Example 4.5.** In Example 3.6 we saw that  $\oplus_{(k)} T^*Q \rightrightarrows Q$  is the poly-symplectic Lie groupoid integrating the trivial  $k$ -poly-Poisson structure on  $Q$ . In this case, for a free and proper  $\mathbb{G}$ -action on  $Q$ , the hamiltonian action of Prop. 4.5 is the one induced by cotangent lift, see Example 4.1(a). We conclude that the poly-symplectic reduction in Theorem 4.6 is  $\oplus_{(k)} T^*(Q/\mathbb{G})$ , as in Example 4.4, which is a presymplectic groupoid integrating the trivial poly-Poisson structure on  $Q/\mathbb{G}$ .

**Example 4.6.** Recall that for a simply connected manifold  $M$ , the  $k$ -poly-symplectic manifold  $(M, \omega)$ , viewed as poly-Poisson manifold, is integrated by the  $s$ -simply connected poly-symplectic groupoid  $M \times M \rightrightarrows M$  endowed with the poly-symplectic form  $t^* \omega - s^* \omega$ , where  $t, s$  are the natural projections from  $M \times M$  to  $M$ . If  $(M, \omega)$  is equipped with a hamiltonian poly-symplectic action of the Lie group  $\mathbb{G}$  and  $J_0 : M \rightarrow \mathfrak{g}_{(k)}^*$  is its moment map, then the moment map (4.15) for the hamiltonian action on the groupoid is  $J = t^* J_0 - s^* J_0$ . If the action on  $M$  is free, proper, reducible and  $0 \in \mathfrak{g}_{(k)}^*$  is a clean value for  $J$ , then the symplectic groupoid  $J^{-1}(0)/\mathbb{G}$  over  $M/\mathbb{G}$  integrates the reduced poly-Poisson structure  $(S_{red}, P_{red})$  induced by  $(M, \omega)$ .

The poly-symplectic groupoid  $\mathcal{G}_{red}$  in Theorem 4.6 is not necessarily the source-simply connected Lie groupoid integrating the reduced structure. This claim is illustrated on [17, Example 4.8] for the case  $k = 1$ .

**Remark 4.7.** Rather than assuming that  $0$  is a clean value of the moment map  $J$  on  $\mathcal{G}$ , one can also proceed as in [7, Prop. 5.3] and consider the source-simply-connected groupoid  $\mathcal{G}_0$  integrating the Lie algebroid  $(J^s)^{-1}(0)$ . With the same arguments as in [7, Prop. 5.3], one can see that this Lie groupoid is equipped with a  $\mathbb{G}$ -action and inherits a  $\mathbb{G}$ -basic multiplicative 2-form  $\omega_0 \in \Omega^2(\mathcal{G}_0, \mathbb{R}^k)$  from the natural map  $\mathcal{G}_0 \rightarrow \mathcal{G}$ , integrating the inclusion  $(J^s)^{-1}(0) \rightarrow S$ . Then  $\mathcal{G}_{0,red} = \mathcal{G}_0/\mathbb{G}$  is a Lie groupoid over  $M/\mathbb{G}$  and  $\omega_0$  reduces to a poly-symplectic form  $\omega_{0,red}$  on  $\mathcal{G}_{0,red}$  integrating the quotient poly-Poisson structure  $(S_{red}, P_{red})$ .

Finally, previous remark allows us to conclude that reduced poly-Poisson structure  $(S_{red}, P_{red})$  is integrable if the Lie algebroid  $(S, P)$  is also integrable.

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